Physics

Lesson Plan #5
A Mathematical Model of Motion
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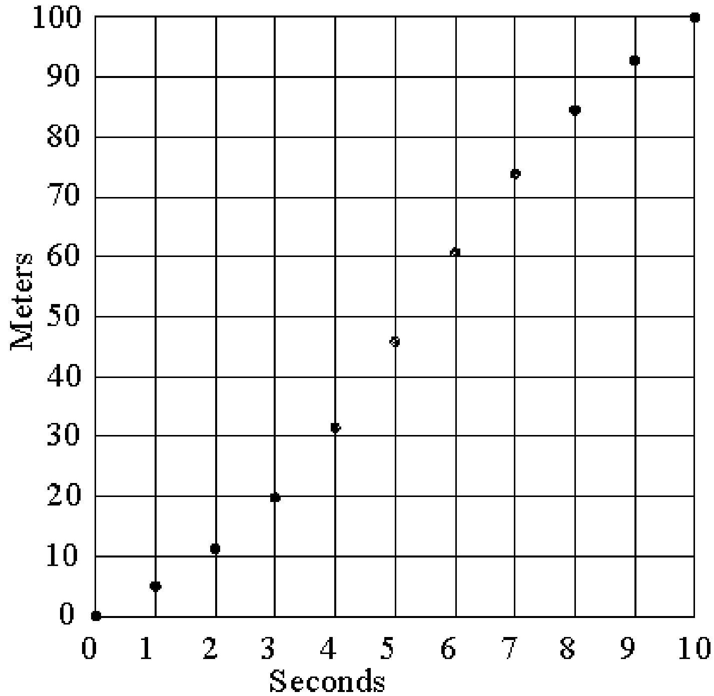
Graphing Motion in One Dimension

Objectives: Interpret graphs of position versus time for a moving object to determine the velocity of the object; Describe in words the information presented in graphs and draw graphs from descriptions of motions; Write equations tat describe the position of an object moving at constant speed

- Position – time Graphs
    - Roller Coasters students have ridden
    - We use position-time graphs to determine where an object is relative to time.
    - If a video camera was to capture a person running a 100 m event, then we could count frames and determine the person's position every second – image this had been done, and here are the results:

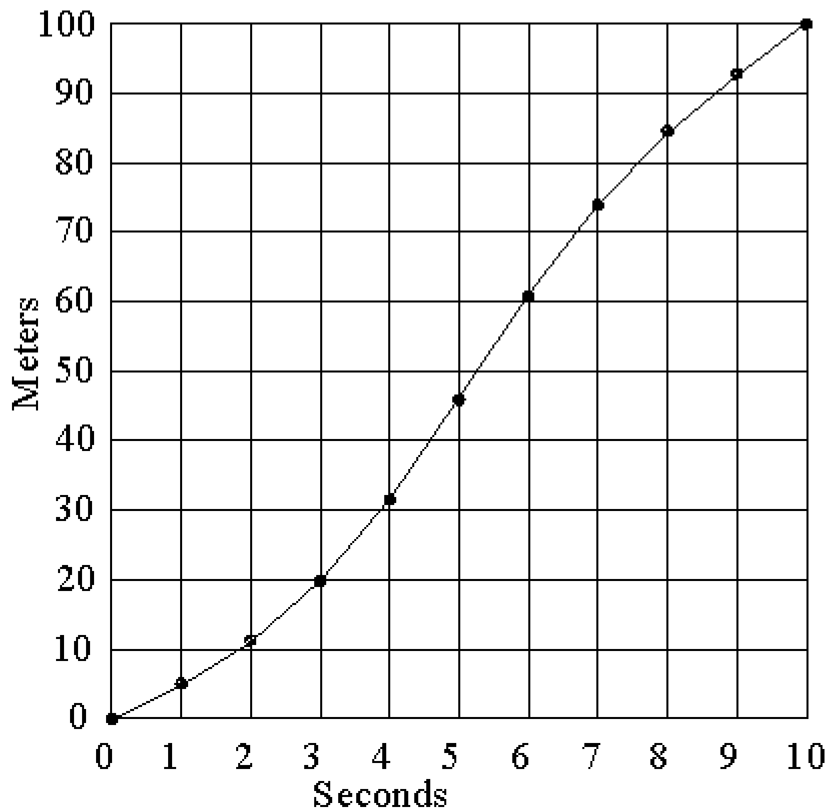
Time	Distance	Time	Distance
0 s	0 m	6 s	60 m
1 s	5 m	7 s	74 m
2 s	11 m	8 s	85 m
3 s	20 m	9 s	93 m
4 s	30 m	10 s	100 m
5 s	45 m		

    - When plotted as Time vs. Distance



p-t Fig. 1

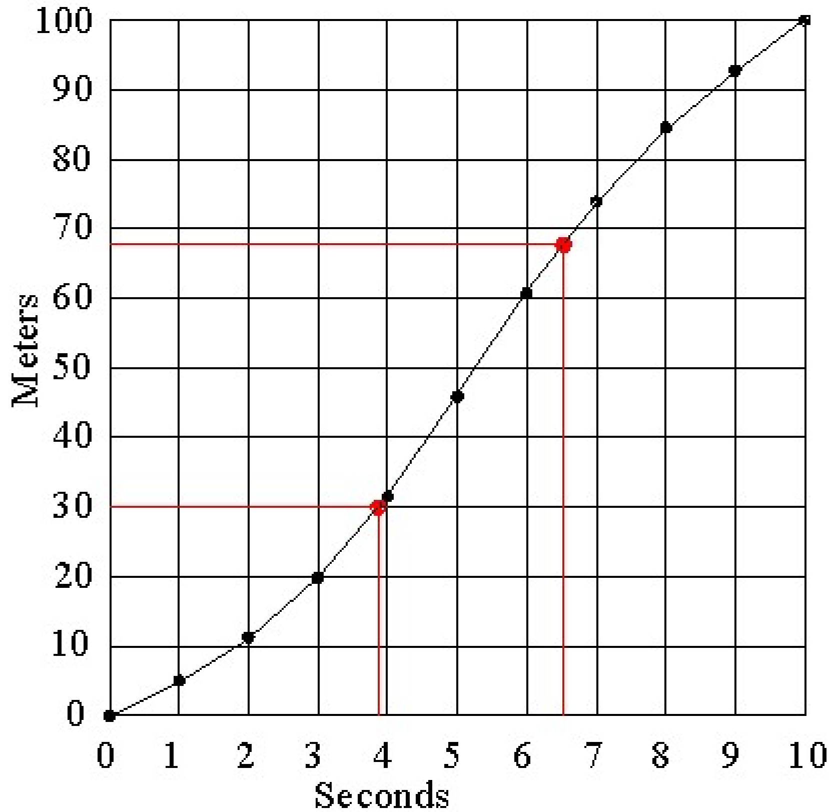
- We now know where the runner was at each second, but how about between seconds?
- By joining the dots together we get a best guess as to the position between know positions



p-t Fig. 2

- When locating the runner's position we have to ask "how long was he at any location?"
  - An instant? – if an instant is any finite amount of time, then since there was change of position during that time, the runner was at rest – thus we define "instantaneous" as a non-finite period of time. Typically use *x* to represent instantaneous position at a particular instant of time

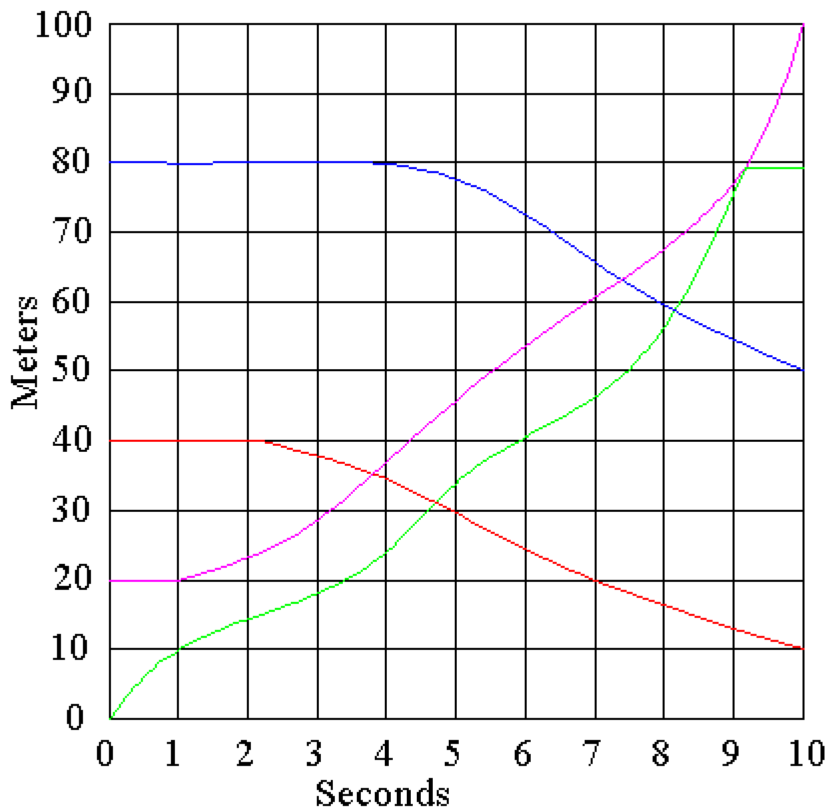
- Data from a Position-Time Graph
  - When did the runner reach the 30 meter mark?
    - Restate – At what time was the position of the object equal to 30m?
  - Where was he after 6.5 seconds?
    - Restate – What was the position of the object at 6.5s?



p-t Fig. 3

- For what time was the position equal to the 30 meter mark, find 30 m on the meter scale and go horizontally to the curve – where this intersects the curve draw a line vertically down – where this line intersects the time scale is the answer – 3.9s
- For what was the position of the object at 6.5s, find 6.5s on the time scale, draw a line vertically until you intersect the curve, then draw a line horizontally to the meter scale – where the line intersects the meter scale is the position of the object at 6.5 s (68 m)

- Graphing two or more objects
  - Multiple objects can be plotted on one graph

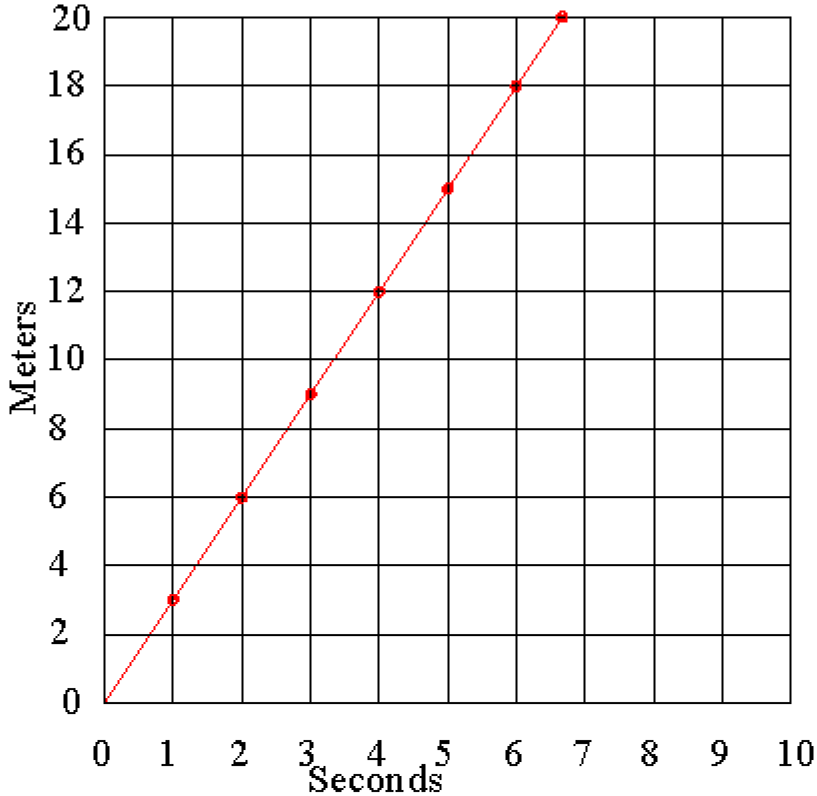


p-t Fig. 4

- In this Position-Time graph of a T-rex on his way home to his mate (starts at 0,0) he is stalked by a trio of velocoraptors.
  - Describe the motion for each velocoraptors
    - The first (at top) waits at a position 80 m from the T-Rex for 4 s and then intercepts the T-Rex at 58 m from the T-Rex starting position – so he was moving in a negative direction, with a negative velocity.
    - The second (middle) waits at a position 40 m from the T-rex for 2.5 seconds before intercepting the T-Rex at 4.5 s and a position of 29 m from the T-Rex's starting position – so he was moving in a negative direction, with a negative velocity.
    - The third (bottom) waits at a position 20 m from the start of the T-Rex for 1 s before starting his motion. He intercepts the T-Rex at 9.2s at a position 79 m from where the T-Rex started and then continues on.
    - The T-Rex moves no farther than 79 m indicating that T-Rex could be injured or dead

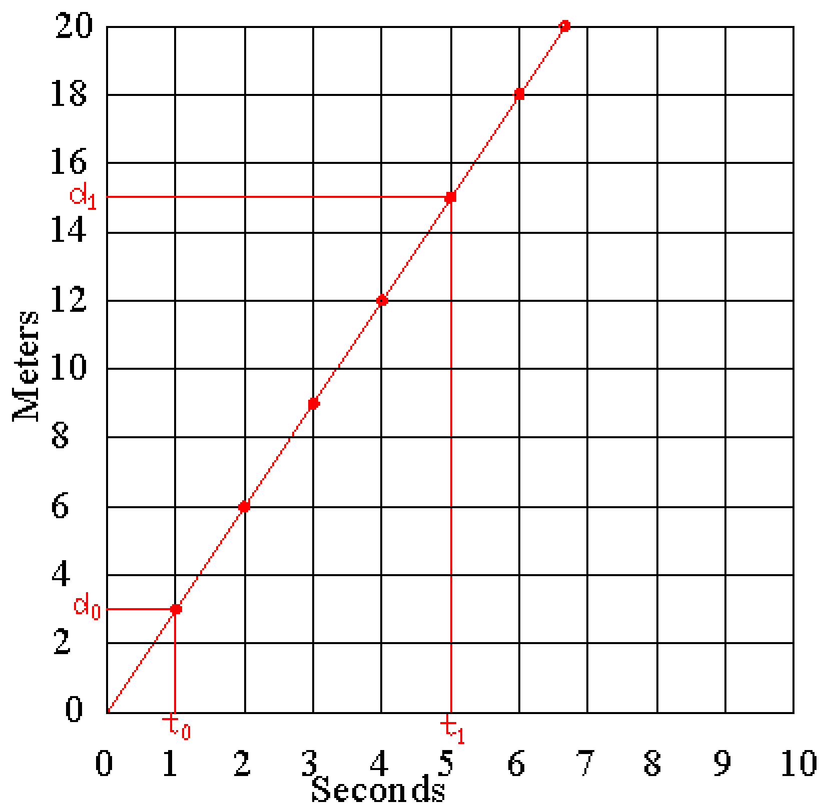
- Uniform Motion
  - Rounding a mark, a sailboat travels 3 m in a straight line for the first second, and then 3 m in the next second and so on

- Uniform Motion means equal displacements have occurred during successive equal time intervals



p-t Fig. 5

- The line drawn through the points represents the position of the sailboat each second is a straight line
- Slope of the line can reveal information
  - Slope = $\frac{rise}{run}$
  - By taking any two points on the line we can determine the slope of the line



p-t Fig. 6

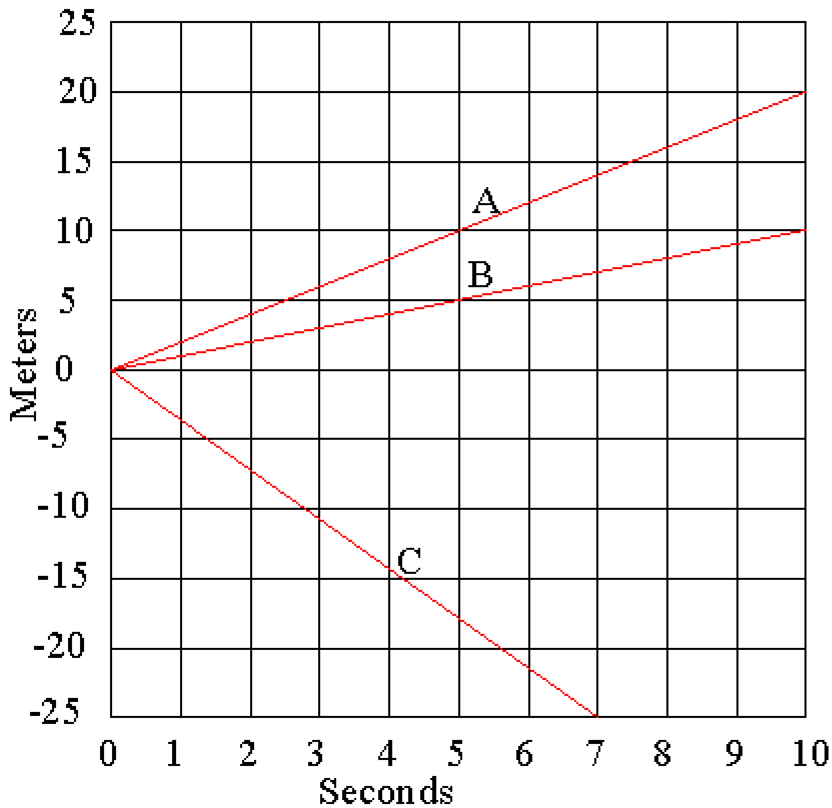
- In this case Slope = $\frac{\Delta d}{\Delta t} = \frac{d_1 - d_0}{t_1 - t_0}$
- Look familiar $\overline{v} = \frac{\Delta d}{\Delta t} = \frac{d_1 - d_0}{t_1 - t_0}$, so the slope of the line is also the average velocity of the boat
- Taking Δd for any corresponding Δt would give us the same average velocity on a straight line
- Note that $\frac{d}{t}$ is not the average velocity of the boat
  - For d = 11 m and t = 3.5 s you would get 11/3.5 = 3.3 m/s

- Uniform Motion, Multiple Objects, & Velocity
  - Look at the Position-Time graph of 3 sailboats at a turning mark in a race. From the P-T graph, what can you tell about the average velocity of each boat and the position of each boat?
  - A is moving at average of $\frac{\Delta d}{\Delta t} = \frac{d_1 - d_0}{t_1 - t_0} = \frac{20m - 0m}{10s - 0s} = 2m/s$
  - B is moving at average of $\frac{\Delta d}{\Delta t} = \frac{d_1 - d_0}{t_1 - t_0} = \frac{10m - 0m}{10s - 0s} = 1m/s$

- C is moving at average of

$$\frac{\Delta d}{\Delta t} = \frac{d_1 - d_0}{t_1 - t_0} = \frac{-25m - 0m}{7s - 0s} = -3.6m/s$$

  - The – velocity indicates that C is moving in the negative direction
  - A downward sloping line has a negative slope indicating negative velocity
  - Negative velocity indicates motion in the opposite direction



p-t Fig. 7

- When and Where by using an equation
  - We have been able to determine position using a graphical method, but it is possible to do the same algebraically
  - Remember that $\overline{v} = \frac{\Delta d}{\Delta t} = \frac{d_1 - d_0}{t_1 - t_0}$
  - Also remember from the last lessons that all algebra is done using the components of vectors and not the vectors themselves.
  - If we choose the origin where t_0 = 0s, then we would be left with

    $$\overline{v} = \frac{d_1 - d_0}{t_1} \triangleright d_1 = d_0 + \overline{v}t_1$$

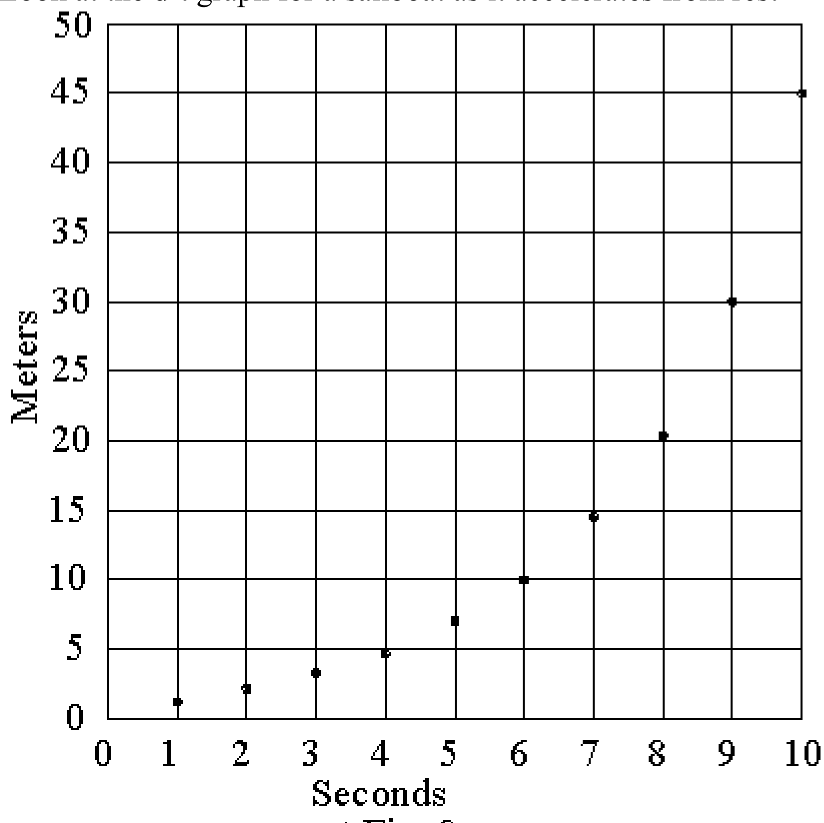
  - We can generalize even more by:

- Letting t be any value of t_1 and d be the position at that time
- Since we are working with constant velocity then average velocity = instantaneous velocity
- So v represents velocity, d_0 represents the position at t_0, we can write $d = d_0 + vt$ - solving for the position of an object moving at constant velocity
  - Equation has 4 variables
    - Initial position = d_0
    - Constant velocity = v
    - Time = t
    - Position at time $t = d$
- Knowing any three, you can solve for the fourth

- Graphing Velocity in One Dimension

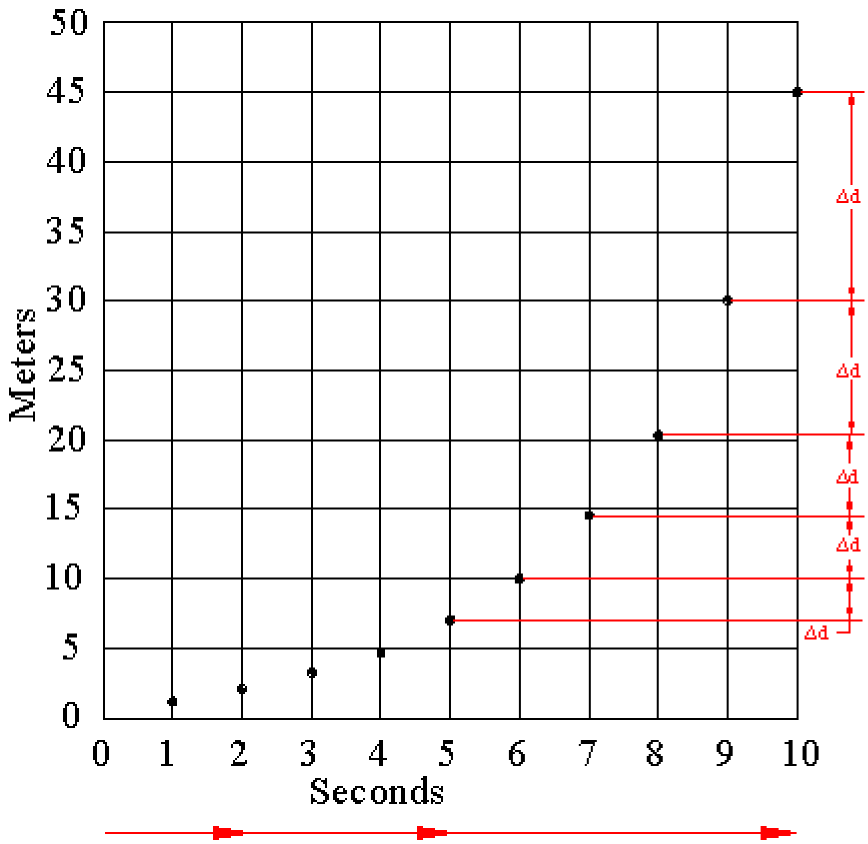
Objectives: Determine from a graph of velocity vs. time, the velocity of an object at a specific time; Interpret a v-t graph to find the time at which an object has a specific velocity; Calculate the displacement of an object form a v-t curve.

- Determining Instantaneous Velocity
  - Look at the d-t graph for a sailboat as it accelerates from rest



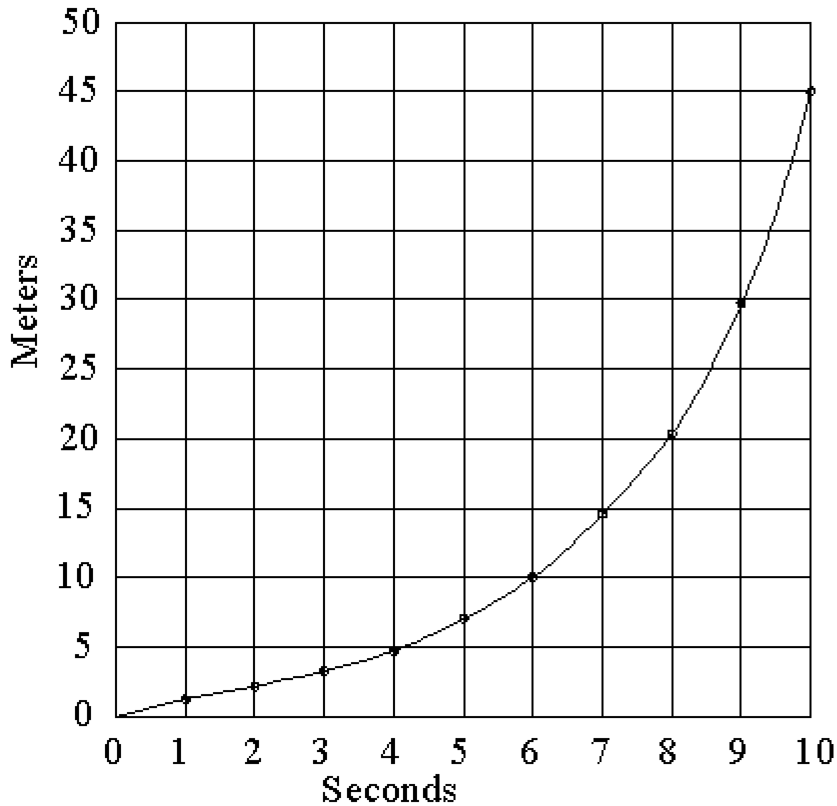
p-t Fig. 8

- What we see is that for each interval of time that the distance traveled increases – therefore the sailboat is going faster and faster
  - Looking at it with a displacement vectors



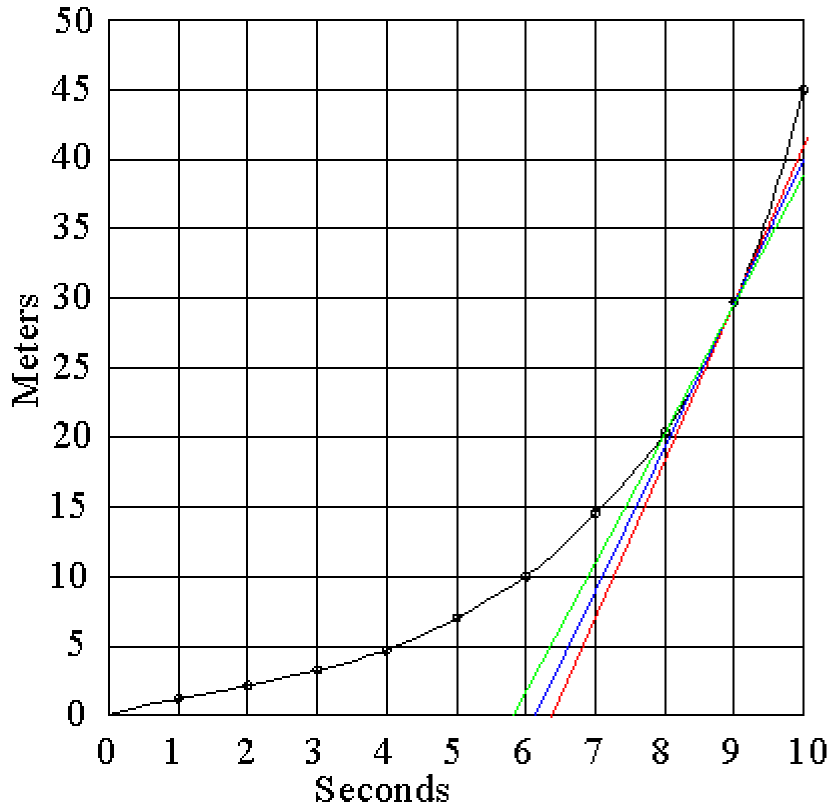
p-t Fig. 9

- So how fast was the sailboat going between the 8th and 9th seconds?
  - We can determine the average velocity by drawing a line between two points and taking the slope of that line
  - We can see that the distance went from 21 meters to 30 meters in one second, so the average velocity would be $\frac{\Delta d}{\Delta t} = \frac{30m - 21m}{9s - 8s} = \frac{9m}{1s} = 9m/s$
  - The velocity probably changed during the time from the 8th to the 9th second, so se need more data – first let us draw a best fit curve



p-t Fig. 10

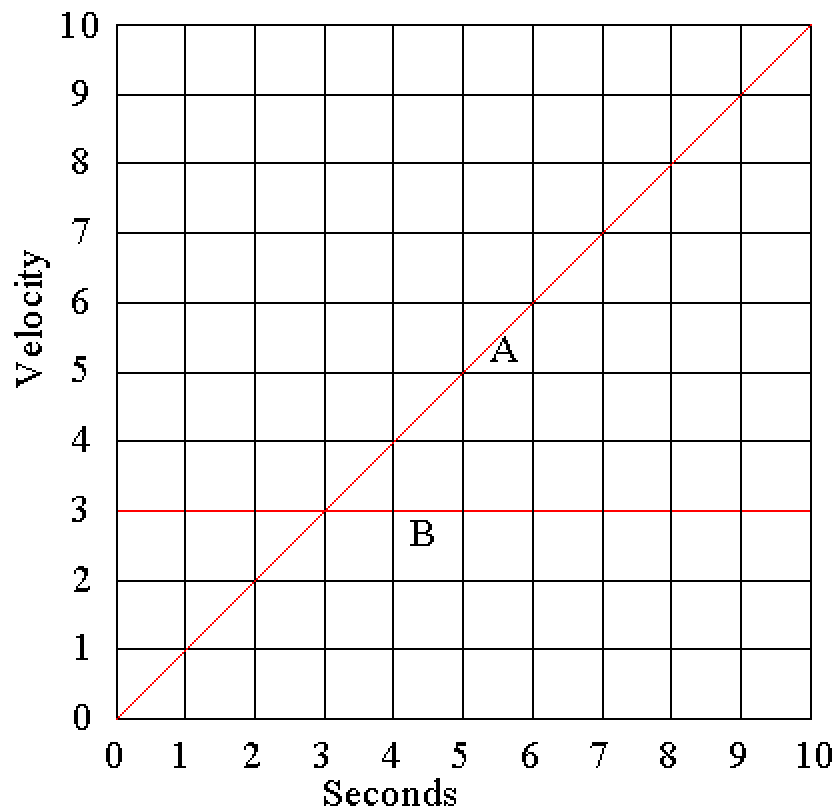
- If we took the average velocity from 8.5 s to 9s then we would have $\frac{\Delta d}{\Delta t} = \frac{30m - 25m}{9s - 8.5s} = \frac{5m}{.5s} = 10m/s$
- If we took the average velocity from 8.75 s to 9s then we would have $\frac{\Delta d}{\Delta t} = \frac{30m - 27.5m}{9s - 8.75s} = \frac{2.5m}{.25s} = 10.0m/s$



p-t Fig. 11

- As the time interval becomes smaller and smaller we get closer and closer to the actual velocity at a moment in time – the instantaneous velocity.
- The instantaneous velocity would be the slope of a line tangent to a point on the curve

- Velocity-Time Graphs
  - With an understanding of velocity at an instant of time - instantaneous velocity - we can now plot velocity vs. time
    - Below is the plot of two sailboats – one maintaining a steady 3 m/s, while the other sailboat is accelerating



p-t Fig. 12

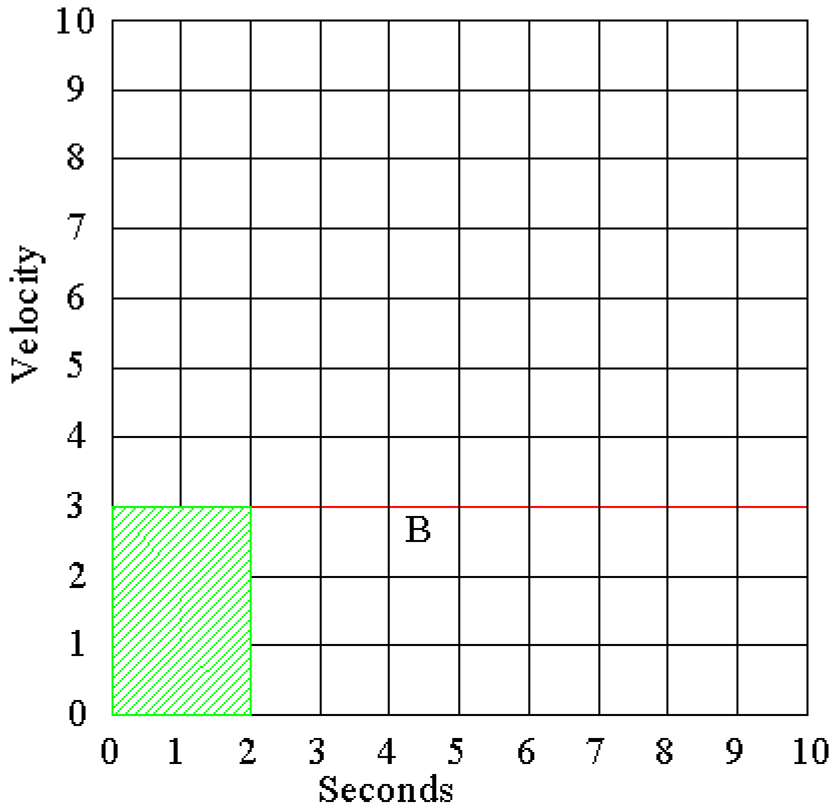
- Which line represents the sailboat with uniform motion (B)
  - Uniform motion on a v-t graph is a horizontal line
  - Slope = 0
- Which line represents the sailboat that is accelerating (A)
  - Change of velocity , has slope not equal to 0
- If the sailboat with constant velocity were going in the opposite direction, how would you graph it?
  - If coordinates were kept the same, then it would be a horizontal line below the t axis

o In the d-t graphs we saw we could determine position at a point in time – what can we see from the v-t graph?
  - Boats cross paths at 3 seconds each with a velocity of 3 m/s
  - Does not mean they were in the same position at this point
  - v-t graphs do not give information about position, but can yield displacement as we will see

\- Displacement from a v-t graph

o For an object moving at constant velocity
  - $v = \bar{v} = \dfrac{\Delta d}{\Delta t}$, so Δd = v/ Δt
  - In the graph below, we see that v is the height of the curve (straight line in this case) above the t-axis, while Δt is the width of the shaded rectangle.

- The area of the rectangle is v Δt or Δd – in other words, the displacement is the area under the v-t curve



p-t Fig. 13

- Acceleration

Objectives: Determine form the curves on a velocity-time graph both the constant and instantaneous acceleration; Determine the sign if acceleration using a v-t graph and a motion diagram; Calculate the velocity and the displacement of an object undergoing constant acceleration.

- Determining Average Acceleration
    - Using the graph below of two airplanes determine the average acceleration
        - $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_1 - v_0}{t_1 - t_0}$
        - As with d-t graphs, the slope of the line in a v-t graph is meaningful – for v-t graphs slope is the average velocity
        - In the graph below we see that the slope of sailboat B is 0 so we can state that the average acceleration is 0 as well
        - In the case of sailboat A, we see that when v_0=0 m/s that t_0 = 0s and when v_1=10 m/s that t_1=10 s, then average acceleration is 10 m/s / 10s = 1 m/s/s or 1 m/s^2
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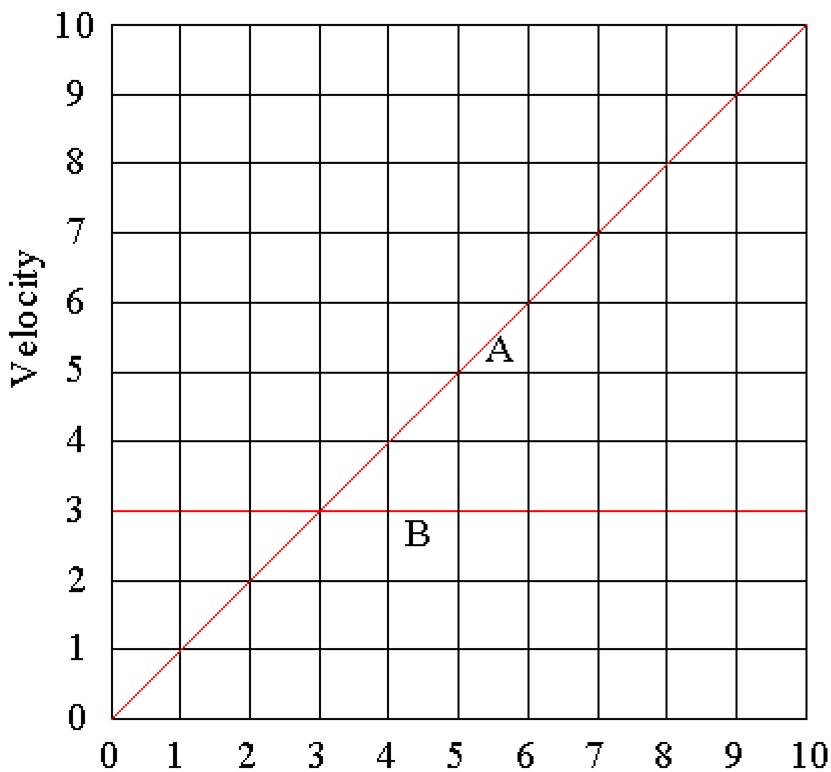
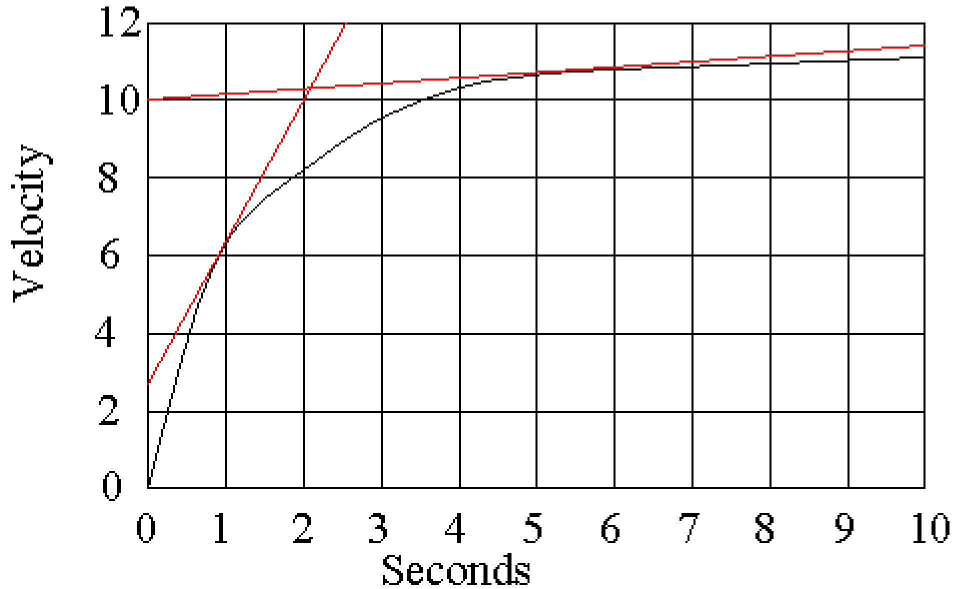


p-t Fig. 12

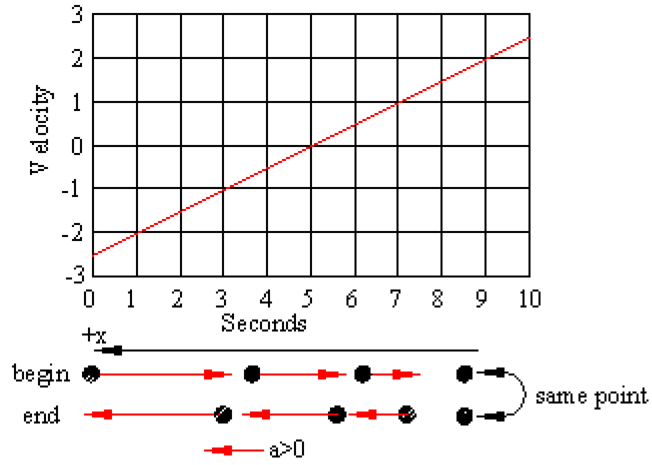
- Recall that when a object has constant velocity or uniform motion, the slope of the p-t graph is constant
- In the above graph, the slope of sailboat A's velocity is constant, therefore we state that sailboat A has **constant acceleration**
- With constant velocity, we stated that the average velocity was the same as the instantaneous velocity, but with changing velocity we could find the instantaneous velocity by plotting the tangent on the d-t graph
- The same holds true on a v-t graph – when the acceleration is not constant, we can find the instantaneous acceleration by finding the tangent of the v-t graph at a particular instant of time
- Take a look at graph below of a sprinter on the 100 m dash – how would you describe the sprinter's velocity and acceleration?
    - Sprinter starts at 0 m/s, increases rapidly for a few seconds, then when he gets to about 10 m/s, velocity becomes almost constant
    - If we examine the graph at 1 second and 5 seconds:
        - Draw tangent line at 1 second
            - $a = \frac{rise}{run} = \frac{12.0m/s - 3.0m/s}{2.5s - 0.0s} = 3.6m/s^2$
        - Draw tangent line at 5 second

- $a = \frac{rise}{run} = \frac{10.7m/s - 10.0m/s}{10s - 0.0s} = .076m/s^2$

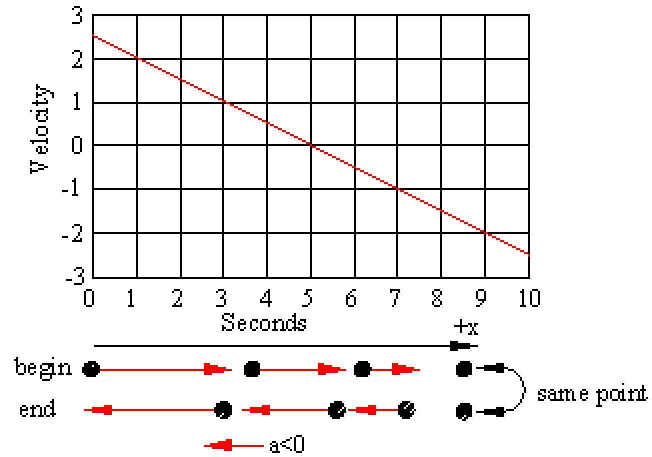


p-t Fig. 14

- Positive and Negative Acceleration
  - The above examples of the sailboat and sprinter were for one that was not changing its velocity and another that was increasing it's velocity – a positive acceleration –
  - Consider a ball being rolled up a slanting driveway,
    - What happens?
      - Ball first rolls up the hill, slowing down, stops briefly, then rolls back down the hill at an increasing speed.
      - Take a look at the two graphs below
      - We see that the ball starts out with a large velocity vector that decreases in time, until it reaches 0
      - Then the velocity vector starts increasing, in the opposite direction until we reach the end
      - In fig 16 note that the acceleration is positive when the ball is heading up the slope and negative when going down the slope. This due to having defined where the ball started as above the origin
      - In fig 15 note that the acceleration is positive when the ball is heading down the slope and negative when going up the slope. This due to having defined where the ball started as below the origin.

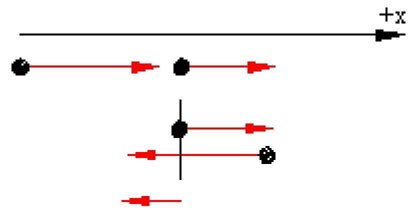


p-t Fig 16



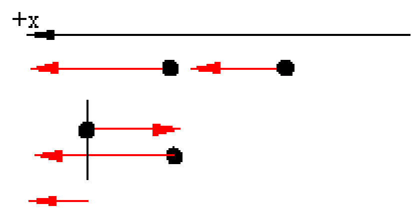
p-t Fig 15

- Acceleration when instantaneous velocity is 0
  - What is the acceleration of the ball when it stops and reverses direction?
    - The answer is in fig 15 & 16 – acceleration is always in the same direction
    - In fig 15 & 16 there is a line under the graph representing direction of motion (not velocity)
    - In fig 15, the ball first starts moving in the direction of increasing x, therefore the sign of the velocity is positive as well – but what about acceleration?
      - Subtract an earlier velocity vector from a later one
      - The acceleration always points down hill (due to gravity) and since it points in the opposite direction of the motion axis, it is negative as shown in fig 17



p-t Fig. 17

- In fig 16, the ball first starts moving in the direction of decreasing x, therefore the sign of the velocity is negative – but what about acceleration?
    - Subtract an earlier velocity vector from a later one
    - The acceleration always points down hill (due to gravity) and since it points in the same direction of the motion axis, it is positive as shown in fig 18



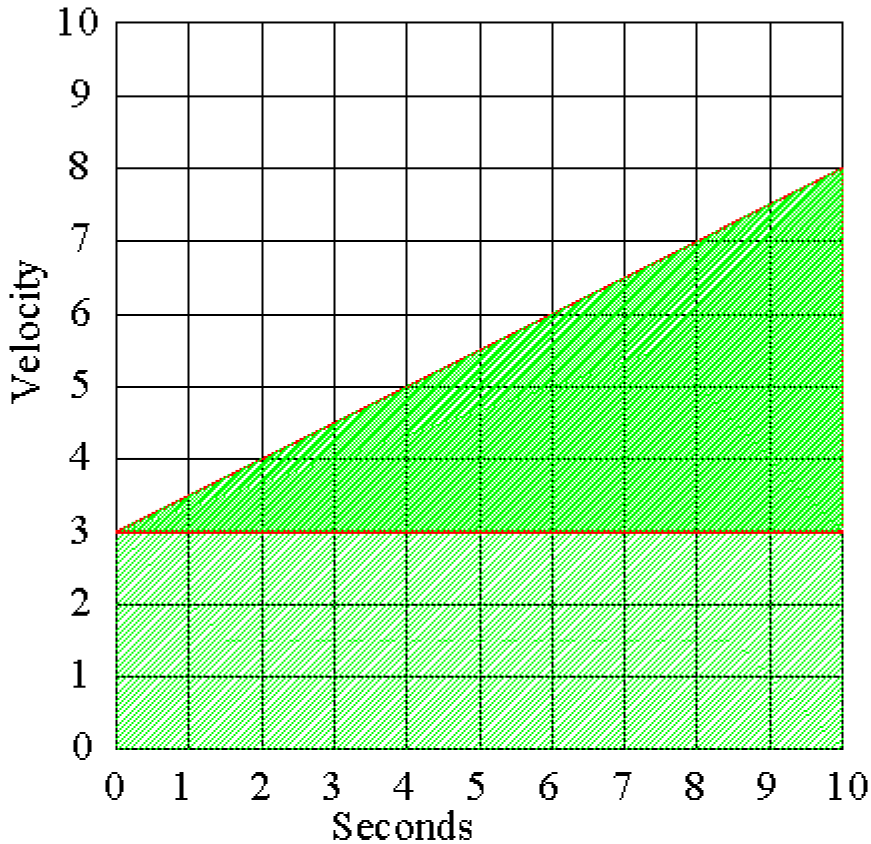
p-t Fig. 18

- What to remember is that when the ball's velocity reaches 0, it is 0 at an instant of time, not for an instant of time – the time the velocity of the ball is zero is zero.

- Calculating Velocity from Acceleration
    - With constant velocity we found we could find the position of an object – in the same way we can rearrange the definition of average acceleration to find the velocity of an object
    - $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_1 - v_0}{t_1 - t_0}$, once again assuming that $t_0 = 0$, and letting t be any value of t_1 and v be the value of the velocity at that time. And since we are only considering constant acceleration then $\bar{a} = a$
    - Substituting and rearranging
        - $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_1 - v_0}{t_1 - t_0} \longrightarrow a = \frac{v - v_0}{t} \longrightarrow v = v_0 + at$
        - This equation is good for obtaining the velocity (v) of an object under constant acceleration, to find the time (t) at which an object under constant

acceleration has a given velocity, or given both the velocity (v) and the time (t) it occurred you can calculate the initial velocity

- Displacement Under Constant Acceleration
  - We used the area under the curve in a v-t graph to calculate the displacement of an object when the velocity was constant – we can do the same for a v-t graph of an object under constant acceleration



p-t Fig. 19

- When we calculated the displacement for constant velocity we found the area of the rectangle for the time we were interested in – here we do the same
  - If the velocity had been constant, the we would simply find the area under that line (3 m/s in this case) as represented by the lighter shade of green
  - Instead, the velocity increased from 3 m/s to 8 m/s and that is from v_0 to v, thus the displacement is increased by the area of the triangle $\frac{1}{2}(v-v_0)t$
  - The total displacement is then $d = v_0 t + 1/2(v - v_0)t$, combining terms we get $d = 1/2(v + v_0)t$

- If the initial position of d is not 0, then we must add d_0, $d = d_0 + 1/2(v + v_0)t$
- Frequently the velocity at time t is not known, but because $v = v_0 + at$ we can substitute $v_0 + at$ for v and obtain $d = d_0 + 1/2(v_0 + v_0 + at)t$ which simplifies to $d = d_0 + v_0 t + 1/2at^2$
- So far we have equations that involves position, velocity and time but not acceleration, $d = d_0 + 1/2(v + v_0)t$, and an equation that involves position, acceleration and time, but not velocity, $d = d_0 + v_0 t + 1/2at^2$, can we come up with a formula that that relates position, velocity and acceleration, but not time?
    - Start with $d = d_0 + 1/2(v + v_0)t$ and $v = v_0 + at$
    - Solving $v = v_0 + at$ for t we get $t = \frac{v - v_0}{a}$
    - Substitute that into $d = d_0 + 1/2(v + v_0)t$ and we get $d = d_0 + 1/2(v_0 + v)(\frac{v - v_0}{a})$
    - Solving for v we get $v^2 = {v_0}^2 + 2a(d - d_0)$

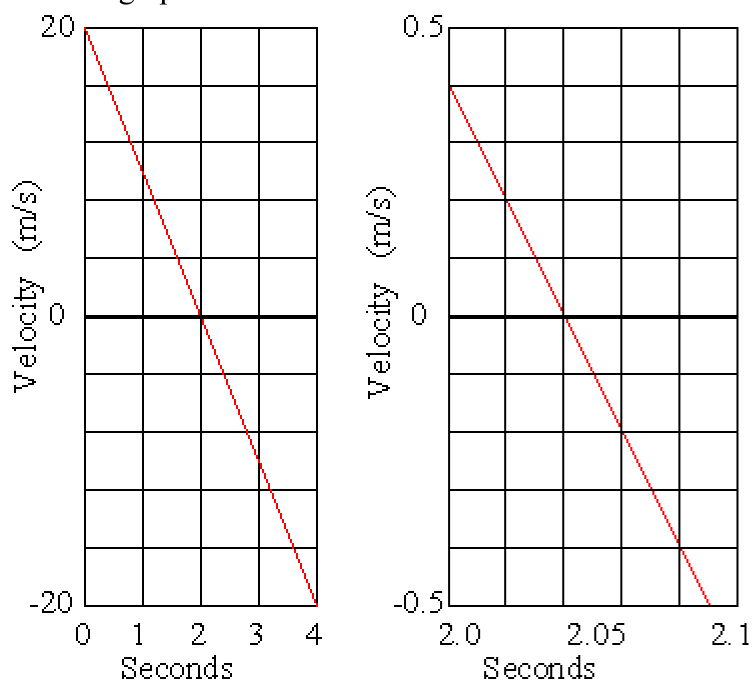
Equations for Uniform Motion		
Equation	Variables	Initial Conditions
$v = v_0 + at$	t v a	v_0
$d = d_0 + 1/2(v + v_0)t$	t d v	d_0 v_0
$d = d_0 + v_0 t + 1/2at^2$	t d a	d_0 v_0
$v^2 = {v_0}^2 + 2a(d - d_0)$	d v a	d_0 v_0
	d v a	d_0 v_0

Free Fall

Objectives: Recognize the meaning of the acceleration due to gravity; Define the magnitude of the acceleration due to gravity as a positive quantity and determine the sign of the acceleration relative to the chosen coordinate system; Use the motion equations to solve problems involving freely falling objects.

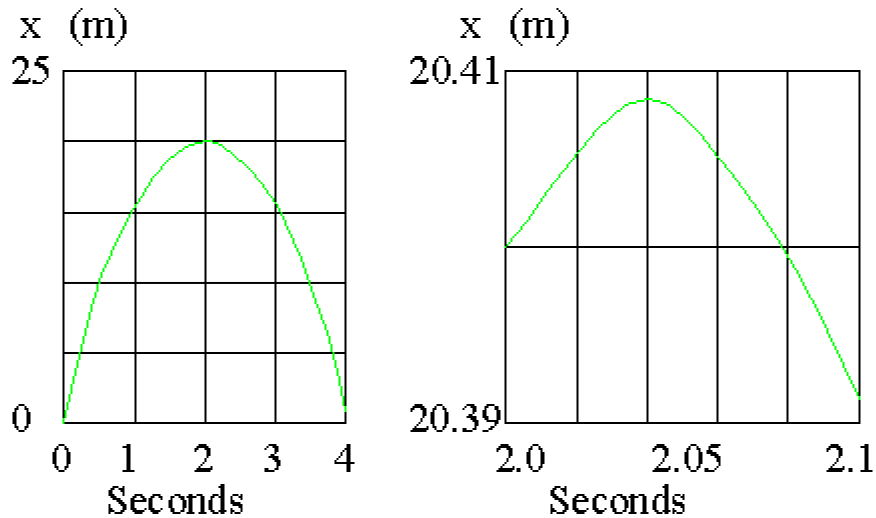
- Acceleration due to Gravity
    - Drop a piece of paper – ball it up and drop it – what is the difference?
    - Galileo recognized over 400 years ago that in order to study the motion of falling bodies, air resistance had to be ignored
        - He used inclined planes to reduce the effect of gravity so that he could make careful measurements with his simple instruments

        - He found that, neglecting air resistance, that all freely falling objects had the same rate of acceleration
            - No matter the mass or height they were dropped from, whether they were dropped or thrown
    - Acceleration due to gravity has a special symbol *g* and is equal to 9.80 m/s^2
        - We now know that there are small variations of *g* at different places on the earth, but 9.80 m/s^2 is the accepted average value
- *g* is a positive value
    - *g* is a magnitude of acceleration
    - *g* may act in a negative direction, depending on the chosen coordinate system
        - If up is chosen to be positive, then the acceleration due to gravity is negative
        - If down is chosen to be positive, then the acceleration due to gravity is positive
- What happens when you throw a ball up at 20 m/s?
    - If we take up to be positive, then $a = -g = -9.80\text{m/s}^2$ then at the end of 1s the velocity will have been reduced by 9.8 m/s^2 to 10.2 m/s
    - After 2s the velocity would be 0.4 m/s, just barely moving upward
    - After 3s the velocity would be –9.4 m/s, the ball has reached the top of its travel and started back down
    - After 4s the velocity is –19.2 m/s meaning it has almost reached the position from which it started
    - A v-t graph of the event



p-t Fig. 20

- A d-t graph of the same event show us



p-t Fig. 21

- The ball has it's maximum height when it velocity reaches 0